

OPTIMAL QUANTIZERS FOR PROBABILITY DISTRIBUTIONS ON NONHOMOGENEOUS R-TRIANGLES

MRINAL KANTI ROYCHOWDHURY

ABSTRACT. Quantization of a probability distribution refers to the idea of estimating a given probability by a discrete probability supported by a finite set. In this paper, we have considered a Borel probability measure P on \mathbb{R}^2 which has support the R-triangle generated by a set of three contractive similarity mappings on \mathbb{R}^2 . For this probability measure, the optimal sets of n -means and the n th quantization error are determined for all $n \geq 2$.

1. INTRODUCTION

Optimal quantization is a fundamental problem in signal processing, data compression and information theory. We refer to [GG, GN, Z] for surveys on the subject and comprehensive lists of references to the literature, see also [AW, GKL, GL1]. For mathematical treatment of quantization one is referred to Graf-Luschgy's book (see [GL2]). For most recent work about quantization for uniform distribution interested readers can also see [DR]. Let \mathbb{R}^d denote the d -dimensional Euclidean space, $\|\cdot\|$ denote the Euclidean norm on \mathbb{R}^d for any $d \geq 1$, and $n \in \mathbb{N}$. Then, the n th *quantization error* for a Borel probability measure P on \mathbb{R}^d is defined by

$$V_n := V_n(P) = \inf \left\{ \int \min_{a \in \alpha} \|x - a\|^2 dP(x) : \alpha \subset \mathbb{R}^d, \text{card}(\alpha) \leq n \right\},$$

where the infimum is taken over all subsets α of \mathbb{R}^d with $\text{card}(\alpha) \leq n$. If $\int \|x\|^2 dP(x) < \infty$ then there is some set α for which the infimum is achieved (see [AW, GKL, GL1, GL2]). Such a set α for which the infimum occurs and contains no more than n points is called an *optimal set of n -means*, or *optimal set of n -quantizers*. The elements of an optimal set are called *optimal points*. The collection of all optimal sets of n -means for a probability measure P is denoted by $\mathcal{C}_n := \mathcal{C}_n(P)$. If α is a finite set, in general, the error $\int \min_{a \in \alpha} \|x - a\|^2 dP(x)$ is often referred to as the *cost* or *distortion error* for α , and is denoted by $V(P; \alpha)$. Thus, $V_n := V_n(P) = \inf \{V(P; \alpha) : \alpha \subset \mathbb{R}^d, \text{card}(\alpha) \leq n\}$. It is known that for a continuous probability measure an optimal set of n -means always has exactly n -elements (see [GL2]). The number

$$\lim_{n \rightarrow \infty} \frac{2 \log n}{-\log V_n(P)},$$

if it exists, is called the *quantization dimension* of the probability measure P . Quantization dimension measures the speed at which the specified measure of the error tends to zero as n approaches to infinity. Given a finite subset $\alpha \subset \mathbb{R}^d$, the *Voronoi region* generated by $a \in \alpha$ is defined by

$$M(a|\alpha) = \{x \in \mathbb{R}^d : \|x - a\| = \min_{b \in \alpha} \|x - b\|\}$$

i.e., the Voronoi region generated by $a \in \alpha$ is the set of all points x in \mathbb{R}^d such that a is a nearest point to x in α , and the set $\{M(a|\alpha) : a \in \alpha\}$ is called the *Voronoi diagram* or *Voronoi tessellation* of \mathbb{R}^d with respect to α . A Voronoi tessellation is called a *centroidal*

2010 *Mathematics Subject Classification.* 60Exx, 28A80, 94A34.

Key words and phrases. Optimal quantizers, quantization error, probability distribution, R-triangle.

The research of the author was supported by U.S. National Security Agency (NSA) Grant H98230-14-1-0320.

Voronoi tessellation (CVT), if the generators of the tessellation are also the centroids of their own Voronoi regions with respect to the probability measure P . A Borel measurable partition $\{A_a : a \in \alpha\}$, where α is an index set, of \mathbb{R}^d is called a Voronoi partition of \mathbb{R}^d if $A_a \subset M(a|\alpha)$ for every $a \in \alpha$. Let us now state the following proposition (see [GG, GL2]):

Proposition 1.1. *Let α be an optimal set of n -means and $a \in \alpha$. Then,*

(i) $P(M(a|\alpha)) > 0$, (ii) $P(\partial M(a|\alpha)) = 0$, (iii) $a = E(X : X \in M(a|\alpha))$, and (iv) P -almost surely the set $\{M(a|\alpha) : a \in \alpha\}$ forms a Voronoi partition of \mathbb{R}^d .

Let α be an optimal set of n -means and $a \in \alpha$, then by Proposition 1.1, we have

$$a = \frac{1}{P(M(a|\alpha))} \int_{M(a|\alpha)} x dP = \frac{\int_{M(a|\alpha)} x dP}{\int_{M(a|\alpha)} dP},$$

which implies that a is the centroid of the Voronoi region $M(a|\alpha)$ associated with the probability measure P (see also [DFG, R]).

Let P be a Borel probability measure on \mathbb{R} given by $P = \frac{1}{2}P \circ S_1^{-1} + \frac{1}{2}P \circ S_2^{-1}$, where $S_1(x) = \frac{1}{3}x$ and $S_2(x) = \frac{1}{3}x + \frac{2}{3}$ for all $x \in \mathbb{R}$. Then, P has support the classical Cantor set C . For this probability measure Graf and Luschgy gave a closed formula to determine the optimal sets of n -means and the n th quantization error for all $n \geq 2$; they also proved that the quantization dimension of this distribution exists and is equal to the Hausdorff dimension $\beta := \log 2 / (\log 3)$ of the Cantor set, but the β -dimensional quantization coefficient does not exist (see [GL3]). Let us now consider a set of three contractive similarity mappings S_1, S_2, S_3 on \mathbb{R}^2 , such that $S_1(x_1, x_2) = \frac{1}{3}(x_1, x_2)$, $S_2(x_1, x_2) = \frac{1}{3}(x_1, x_2) + \frac{2}{3}(1, 0)$, and $S_3(x_1, x_2) = \frac{1}{3}(x_1, x_2) + \frac{2}{3}(\frac{1}{2}, \frac{\sqrt{3}}{2})$ for all $(x_1, x_2) \in \mathbb{R}^2$. Let R be the limit set of these contractive mappings. We call it the *R-triangle* generated by the contractive mappings S_1, S_2, S_3 . Let $P = \frac{1}{3} \sum_{j=1}^3 P \circ S_j^{-1}$. Then, P is a unique Borel probability measure on \mathbb{R}^2 with support the R -triangle generated by S_1, S_2, S_3 . We call it as *R-measure*. For this R -measure, Cómez and Roychowdhury determined the optimal sets of n -means and the n th quantization error for all $n \geq 2$. In addition, they showed that the quantization dimension of the R -measure exists which is equal to one, and it coincides with the Hausdorff dimension of the R -triangle, the Hausdorff and packing dimensions of the R -measure, i.e., all these dimensions are equal to one. Moreover, it was shown that the s -dimensional quantization coefficient for $s = 1$ of the R -measure does not exist (see [CR]).

In this paper, we have considered a set of three contractive similarity mappings S_1, S_2, S_3 on \mathbb{R}^2 , such that $S_1(x_1, x_2) = \frac{1}{4}(x_1, x_2)$, $S_2(x_1, x_2) = \frac{1}{4}(x_1, x_2) + \frac{3}{4}(1, 0)$, and $S_3(x_1, x_2) = \frac{1}{2}(x_1, x_2) + \frac{1}{2}(\frac{1}{2}, \frac{\sqrt{3}}{2})$ for all $(x_1, x_2) \in \mathbb{R}^2$. Let R be the limit set of these contractive mappings. We call it as a *nonhomogeneous R-triangle* generated by the contractive mappings S_1, S_2, S_3 . The term ‘nonhomogeneous’ is used to mean that the basic triangles at each level in the construction of the R -triangle are not of equal shape. Let $P = \frac{1}{5}P \circ S_1^{-1} + \frac{1}{5}P \circ S_2^{-1} + \frac{3}{5}P \circ S_3^{-1}$. Then, P is a unique Borel probability measure on \mathbb{R}^2 with support the nonhomogeneous R -triangle generated by S_1, S_2, S_3 . We call it as *R-measure* or more specifically *nonhomogeneous R-measure*. For this R -measure, in Theorem 3.10, we state and prove an induction formula to determine the optimal sets of n -means for all $n \geq 2$. Once the optimal sets are known, the corresponding quantization errors can easily be obtained. We also give some figures to illustrate the locations of the optimal points (see Figure 1, Figure 2 and Figure 3). In addition, using the induction formula we obtain some results and observations about the optimal sets of n -means which are given in Section 4; a tree diagram of the optimal sets of n -means for a certain range of n is also given (see Figure 4).

2. BASIC DEFINITIONS AND LEMMAS

In this section, we give the basic definitions and lemmas that will be instrumental in our analysis. By a *string* or a *word* ω over an alphabet $I := \{1, 2, 3\}$, we mean a finite sequence $\omega := \omega_1\omega_2\cdots\omega_k$ of symbols from the alphabet, where $k \geq 1$, and k is called the length of the word ω . A word of length zero is called the *empty word*, and is denoted by \emptyset . By I^* we denote the set of all words over the alphabet I of some finite length k including the empty word \emptyset . By $|\omega|$, we denote the length of a word $\omega \in I^*$. For any two words $\omega := \omega_1\omega_2\cdots\omega_k$ and $\tau := \tau_1\tau_2\cdots\tau_\ell$ in I^* , by $\omega\tau := \omega_1\cdots\omega_k\tau_1\cdots\tau_\ell$ we mean the word obtained from the concatenation of ω and τ . As defined in the previous section, the maps $S_i : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ are the generating maps of the R-triangle with similarity ratios s_i for $1 \leq i \leq 3$ respectively, and $P = \sum_{i=1}^3 p_i P \circ S_i^{-1}$ is the probability distribution, where $s_1 = s_2 = \frac{1}{4}$, $s_3 = \frac{1}{2}$, $p_1 = p_2 = \frac{1}{5}$ and $p_3 = \frac{3}{5}$. For $\omega = \omega_1\omega_2\cdots\omega_k \in I^k$, set $S_\omega := S_{\omega_1} \circ S_{\omega_2} \circ \cdots \circ S_{\omega_k}$, $s_\omega := s_{\omega_1}s_{\omega_2}\cdots s_{\omega_k}$ and $p_\omega := p_{\omega_1}p_{\omega_2}\cdots p_{\omega_k}$. Let Δ be the equilateral triangle with vertices $(0, 0)$, $(1, 0)$ and $(\frac{1}{2}, \frac{\sqrt{3}}{2})$. The sets $\{\Delta_\omega : \omega \in I^k\}$ are just the 3^k triangles in the k th level in the construction of the R-triangle. The triangles Δ_{ω_1} , Δ_{ω_2} and Δ_{ω_3} into which Δ_ω is split up at the $(k+1)$ th level are called the *basic triangles* of Δ_ω . The set $R := \bigcap_{k \in \mathbb{N}} \bigcup_{\omega \in I^k} \Delta_\omega$ is the R-triangle and equals the support of the probability measure P . For $\omega = \omega_1\omega_2\cdots\omega_k \in I^k$, let us write $c(\omega) := \#\{i : \omega_i = 3, 1 \leq i \leq k\}$. Then, we have

$$P(\Delta_\omega) = p_\omega = \frac{3^{c(\omega)}}{5^{|\omega|}} \text{ and } s_\omega = \frac{2^{c(\omega)}}{4^{|\omega|}}.$$

Let us now give the following lemma.

Lemma 2.1. *Let $f : \mathbb{R} \rightarrow \mathbb{R}^+$ be Borel measurable and $k \in \mathbb{N}$. Then,*

$$\int f dP = \sum_{\omega \in I^k} p_\omega \int f \circ S_\omega dP.$$

Proof. We know $P = \sum_{i=1}^3 p_i P \circ S_i^{-1}$, and so by induction $P = \sum_{\omega \in I^k} p_\omega P \circ S_\omega^{-1}$, and thus the lemma is yielded. \square

Let $S_{(i1)}$, $S_{(i2)}$ be the horizontal and vertical components of the transformations S_i for $1 \leq i \leq 3$. Then, for any $(x_1, x_2) \in \mathbb{R}^2$ we have $S_{(11)}(x_1) = \frac{1}{4}x_1$, $S_{(12)}(x_2) = \frac{1}{4}x_2$, $S_{(21)}(x_1) = \frac{1}{4}x_1 + \frac{3}{4}$, $S_{(22)}(x_2) = \frac{1}{4}x_2$, $S_{(31)}(x_1) = \frac{1}{2}x_1 + \frac{1}{4}$, and $S_{(32)}(x_2) = \frac{1}{2}x_2 + \frac{\sqrt{3}}{4}$. Let $X := (X_1, X_2)$ be a bivariate continuous random variable with distribution P . Let P_1, P_2 be the marginal distributions of P , i.e., $P_1(A) = P(A \times \mathbb{R}) = P \circ \pi_1^{-1}(A)$ for all $A \in \mathfrak{B}$, and $P_2(B) = P(\mathbb{R} \times B) = P \circ \pi_2^{-1}(B)$ for all $B \in \mathfrak{B}$, where π_1, π_2 are two projection mappings given by $\pi_1(x_1, x_2) = x_1$ and $\pi_2(x_1, x_2) = x_2$ for all $(x_1, x_2) \in \mathbb{R}^2$. Here \mathfrak{B} is the Borel σ -algebra on \mathbb{R} . Then X_1 has distribution P_1 and X_2 has distribution P_2 .

The statement below provides the connection between P and its marginal distributions via the components of the generating maps S_i . The proof is similar to Lemma 2.2 in [CR].

Lemma 2.2. *Let P_1 and P_2 be the marginal distributions of the probability measure P . Then,*

$$\begin{aligned} P_1 &= \frac{1}{5}P_1 \circ S_{(11)}^{-1} + \frac{1}{5}P_1 \circ S_{(21)}^{-1} + \frac{3}{5}P_1 \circ S_{(31)}^{-1} \text{ and} \\ P_2 &= \frac{1}{5}P_2 \circ S_{(12)}^{-1} + \frac{1}{5}P_2 \circ S_{(22)}^{-1} + \frac{3}{5}P_2 \circ S_{(32)}^{-1}. \end{aligned}$$

Lemma 2.3. *Let $E(X)$ and $V(X)$ denote the the expected vector and the expected squared distance of the random variable X . Then,*

$$E(X) = (E(X_1), E(X_2)) = \left(\frac{1}{2}, \frac{\sqrt{3}}{4}\right) \text{ and } V := V(X) = E\|X - \left(\frac{1}{2}, \frac{\sqrt{3}}{4}\right)\|^2 = \frac{27}{176}$$

with $V(X_1) = \frac{3}{44}$ and $V(X_2) = \frac{15}{176}$.

Proof. We have

$$\begin{aligned} E(X_1) &= \int x_1 dP_1 = \frac{1}{5} \int x_1 dP_1 \circ S_{(11)}^{-1} + \frac{1}{5} \int x_1 dP_1 \circ S_{(21)}^{-1} + \frac{3}{5} \int x_1 dP_1 \circ S_{(31)}^{-1} \\ &= \frac{1}{5} \int \frac{1}{4} x_1 dP_1 + \frac{1}{5} \int \left(\frac{1}{4} x_1 + \frac{3}{4}\right) dP_1 + \frac{3}{5} \int \left(\frac{1}{2} x_1 + \frac{1}{4}\right) dP_1, \end{aligned}$$

which implies $E(X_1) = \frac{1}{2}$ and similarly, one can show that $E(X_2) = \frac{\sqrt{3}}{4}$. Now

$$\begin{aligned} E(X_1^2) &= \int x_1^2 dP_1 = \frac{1}{5} \int x_1^2 dP_1 \circ S_{(11)}^{-1} + \frac{1}{5} \int x_1^2 dP_1 \circ S_{(21)}^{-1} + \frac{3}{5} \int x_1^2 dP_1 \circ S_{(31)}^{-1} \\ &= \frac{1}{5} \int \left(\frac{1}{4} x_1\right)^2 dP_1 + \frac{1}{5} \int \left(\frac{1}{4} x_1 + \frac{3}{4}\right)^2 dP_1 + \frac{3}{5} \int \left(\frac{1}{2} x_1 + \frac{1}{4}\right)^2 dP_1 \\ &= \frac{1}{5} \int \left(\frac{1}{16} x_1^2\right) dP_1 + \frac{1}{5} \int \left(\frac{1}{16} x_1^2 + \frac{3}{8} x_1 + \frac{9}{16}\right) dP_1 + \frac{3}{5} \int \left(\frac{1}{4} x_1^2 + \frac{1}{4} x_1 + \frac{1}{16}\right) dP_1 \\ &= \frac{14}{80} E(X_1^2) + \frac{9}{40} E(X_1) + \frac{12}{80} = \frac{14}{80} E(X_1^2) + \frac{21}{80}, \end{aligned}$$

which implies $E(X_1^2) = \frac{7}{22}$. Similarly, one can show that $E(X_2^2) = \frac{3}{11}$. Thus, we see that $V(X_1) = E(X_1^2) - (E(X_1))^2 = \frac{7}{22} - \frac{1}{4} = \frac{3}{44}$, and likewise $V(X_2) = \frac{15}{176}$. Hence,

$$\begin{aligned} E\|X - \left(\frac{1}{2}, \frac{\sqrt{3}}{4}\right)\|^2 &= \iint_{\mathbb{R}^2} \left((x_1 - \frac{1}{2})^2 + (x_2 - \frac{\sqrt{3}}{4})^2\right) dP(x_1, x_2) \\ &= \int (x_1 - \frac{1}{2})^2 dP_1(x_1) + \int (x_2 - \frac{\sqrt{3}}{4})^2 dP_2(x_2) = V(X_1) + V(X_2) = \frac{27}{176}, \end{aligned}$$

which completes the proof of the lemma. \square

Let us now give the following note.

Note 2.4. From Lemma 2.3 it follows that the optimal set of one-mean is the expected vector and the corresponding quantization error is the expected squared distance V of the random variable X . For words $\beta, \gamma, \dots, \delta$ in I^* , by $a(\beta, \gamma, \dots, \delta)$ we mean the conditional expected squared distance of the random variable X given $\Delta_\beta \cup \Delta_\gamma \cup \dots \cup \Delta_\delta$, i.e.,

$$(1) \quad a(\beta, \gamma, \dots, \delta) = E(X | X \in \Delta_\beta \cup \Delta_\gamma \cup \dots \cup \Delta_\delta) = \frac{1}{P(\Delta_\beta \cup \dots \cup \Delta_\delta)} \int_{\Delta_\beta \cup \dots \cup \Delta_\delta} x dP.$$

For $\omega \in I^k$, $k \geq 1$, since $a(\omega) = E(X : X \in J_\omega)$, using Lemma 2.1, we have

$$a(\omega) = \frac{1}{P(\Delta_\omega)} \int_{\Delta_\omega} x dP(x) = \int_{\Delta_\omega} x dP \circ S_\omega^{-1}(x) = \int S_\omega(x) dP(x) = E(S_\omega(X)) = S_\omega\left(\frac{1}{2}, \frac{\sqrt{3}}{4}\right).$$

For any $(a, b) \in \mathbb{R}^2$, $E\|X - (a, b)\|^2 = V + \left\|\left(\frac{1}{2}, \frac{\sqrt{3}}{4}\right) - (a, b)\right\|^2$. In fact, for any $\omega \in I^k$, $k \geq 1$, we have $\int_{\Delta_\omega} \|x - (a, b)\|^2 dP = p_\omega \int \|(x_1, x_2) - (a, b)\|^2 dP \circ S_\omega^{-1}$, which implies

$$(2) \quad \int_{\Delta_\omega} \|x - (a, b)\|^2 dP = p_\omega \left(s_\omega^2 V + \|a(\omega) - (a, b)\|^2 \right).$$

The expressions (1) and (2) are useful to obtain the optimal sets and the corresponding quantization errors with respect to the probability distribution P . Notice that with respect to the median passing through the vertex $(\frac{1}{2}, \frac{\sqrt{3}}{4})$, the R-triangle has the maximum symmetry, i.e., with respect to the line $x_1 = \frac{1}{2}$ the R-triangle is geometrically symmetric. Also, observe that, if the two basic rectangles of similar geometrical shape lie in the opposite sides of the line $x_1 = \frac{1}{2}$, and are equidistant from the line $x_1 = \frac{1}{2}$, then they have the same probability (see Figure 1,

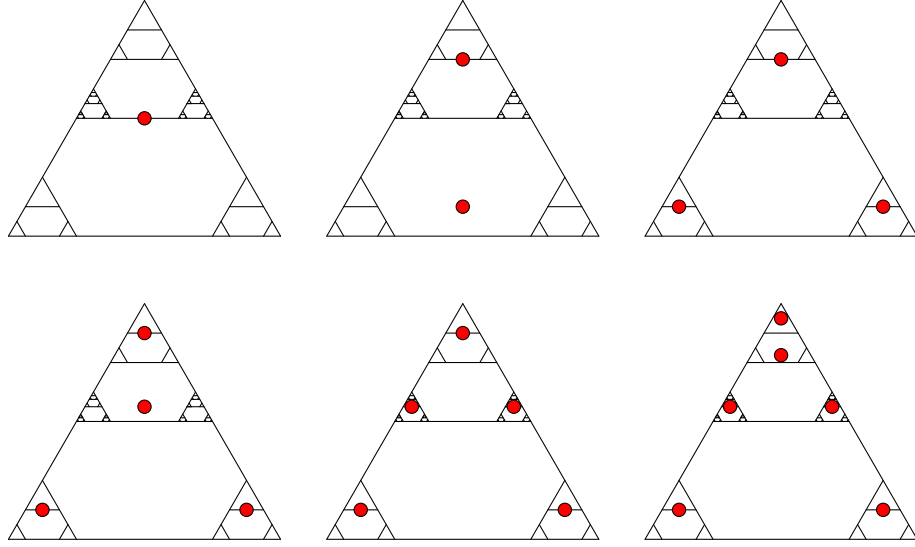
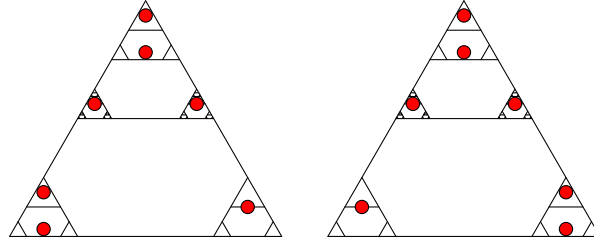
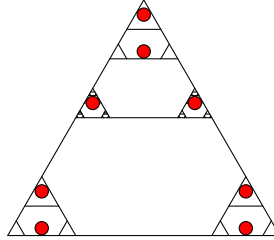
FIGURE 1. Optimal configuration of n points for $1 \leq n \leq 6$.FIGURE 2. Optimal configuration of n points for $n = 7$.FIGURE 3. Optimal configuration of n points for $n = 8$.

Figure 2 or Figure 3); hence, they are symmetric with respect to the probability distribution P as well.

In the next section, we determine the optimal sets of n -means for all $n \geq 2$.

3. OPTIMAL SETS OF n -MEANS FOR ALL $n \geq 2$

In this section let us first prove the following proposition.

Proposition 3.1. *The set $\alpha = \{a(1,2), a(3)\}$, where $a(1,2) = (\frac{1}{2}, \frac{\sqrt{3}}{16})$ and $a(3) = (\frac{1}{2}, \frac{3\sqrt{3}}{8})$, is an optimal set of two-means with quantization error $V_2 = \frac{117}{1408} = 0.0830966$.*

Proof. Let us consider the set of two points β given by $\beta := \{a(1, 2), a(3)\} = \{(\frac{1}{2}, \frac{\sqrt{3}}{16}), (\frac{1}{2}, \frac{3\sqrt{3}}{8})\}$. Then, $\Delta_1 \cup \Delta_2 \subset M((a(1, 2)|\beta))$ and $\Delta_3 \subset M(a(3)|\beta)$, and so the distortion error due to the set β is given by

$$\int \min_{b \in \beta} \|x - b\|^2 dP = \int_{\Delta_1 \cup \Delta_2} \|x - a(1, 2)\|^2 dP + \int_{\Delta_3} \|x - a(3)\|^2 dP = \frac{117}{1408} = 0.0830966.$$

Since V_2 is the quantization error for two-means, we have $V_2 \leq 0.0830966$. Due to maximum symmetry of the R-triangle with respect to the vertical line $x_1 = \frac{1}{2}$, among all the pairs of two points which have the boundaries of the Voronoi regions oblique lines passing through the centroid $(\frac{1}{2}, \frac{\sqrt{3}}{4})$, the two points which have the boundary of the Voronoi regions the vertical line $x_1 = \frac{1}{2}$ will give the smallest distortion error. Let (a, b) and (c, d) be the centroids of the left and right half of the R triangle with respect to the line $x_1 = \frac{1}{2}$. Then, writing $A := \Delta_1 \cup \Delta_{31} \cup \Delta_{331} \cup \Delta_{3331} \cup \dots$ and $B := \Delta_2 \cup \Delta_{32} \cup \Delta_{332} \cup \Delta_{3332} \cup \dots$, we have

$$(a, b) = E(X : X \in A) = (\frac{2}{7}, 0.433013), \text{ and } (c, d) = E(X : X \in B) = (\frac{5}{7}, 0.433013),$$

which yield the distortion error as

$$\int \min_{c \in \{(a, b), (c, d)\}} \|x - c\|^2 dP = \int_A \|x - (a, b)\|^2 dP + \int_B \|x - (c, d)\|^2 dP = \frac{927}{8624} = 0.107491.$$

Notice that $0.107491 > V_2$, and so the line $x_1 = \frac{1}{2}$ can not be the boundary of the two points in an optimal set of two-means, in other words, we can assume that the points in an optimal set of two-points lie on a vertical line. Let $\alpha := \{(p, b_1), (p, b_2)\}$ be an optimal set of two-means with $b_1 \leq b_2$. Since the optimal points are the centroids of their own Voronoi regions, we have $\alpha \subset \Delta$. Moreover, by the properties of centroids, we have

$$(p, b_1)P(M((p, b_1)|\alpha)) + (p, b_2)P(M((p, b_2)|\alpha)) = (\frac{1}{2}, \frac{\sqrt{3}}{4}),$$

which implies $p = \frac{1}{2}$ and $b_1P(M((p, b_1)|\alpha)) + b_2P(M((p, b_2)|\alpha)) = \frac{\sqrt{3}}{4}$. Thus, it follows that the two optimal points are $(\frac{1}{2}, b_1)$ and $(\frac{1}{2}, b_2)$, and they lie in the opposite sides of the point $(\frac{1}{2}, \frac{\sqrt{3}}{4})$, and so we have $\alpha = \{(\frac{1}{2}, b_1), (\frac{1}{2}, b_2)\}$ with $0 < b_1 \leq \frac{\sqrt{3}}{4} \leq b_2 < \frac{\sqrt{3}}{2}$. If the Voronoi region of the point $(\frac{1}{2}, b_2)$ contains points from the region below the line $x_2 = \frac{\sqrt{3}}{8}$, in other words, if it contains points from Δ_1 or Δ_2 , we must have $\frac{1}{2}(b_1 + b_2) < \frac{\sqrt{3}}{8}$ implying $b_1 < \frac{\sqrt{3}}{4} - b_2 \leq 0$, which yields a contradiction. So, we can assume that the Voronoi region of $(\frac{1}{2}, b_2)$ does not contain any point below the line $x_2 = \frac{\sqrt{3}}{8}$. Again, $E(X : X \in \Delta_1 \cup \Delta_2) = (\frac{1}{2}, \frac{\sqrt{3}}{16})$ and $E(X : X \in \Delta_3) = a(3) = (\frac{1}{2}, \frac{3\sqrt{3}}{8})$, and so $\frac{\sqrt{3}}{16} \leq b_1 \leq \frac{\sqrt{3}}{4} < \frac{3\sqrt{3}}{8} \leq b_2 < \frac{\sqrt{3}}{2}$. Notice that $b_1 \leq \frac{\sqrt{3}}{4}$ implies $\frac{1}{2}(b_1 + b_2) \leq \frac{1}{2}(\frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{2}) = \frac{3\sqrt{3}}{8}$, and so $\Delta_{33} \subset M((\frac{1}{2}, b_2)|\alpha)$ yielding $b_2 \leq \frac{7\sqrt{3}}{16}$. Suppose that $\frac{\sqrt{3}}{4} \leq b_2 \leq \frac{5\sqrt{3}}{16}$. Then, if $\frac{1}{4} \leq b_1 \leq \frac{\sqrt{3}}{4}$,

$$\begin{aligned} \int \min_{c \in \alpha} \|x - c\|^2 dP &\geq \int_{\Delta_{33} \cup \Delta_{313} \cup \Delta_{323}} \min_{\frac{\sqrt{3}}{4} \leq b_2 \leq \frac{5\sqrt{3}}{16}} \|x - (\frac{1}{2}, b_2)\|^2 dP + \int_{\Delta_1 \cup \Delta_2} \min_{\frac{1}{4} \leq b_1 \leq \frac{\sqrt{3}}{4}} \|x - (\frac{1}{2}, b_1)\|^2 dP \\ &\geq 0.0937031 > V_2, \end{aligned}$$

which is a contradiction, and if $\frac{\sqrt{3}}{16} \leq b_1 \leq \frac{1}{4}$, then

$$\begin{aligned} \int \min_{c \in \alpha} \|x - c\|^2 dP &\geq \int_{\Delta_3} \min_{\frac{\sqrt{3}}{16} \leq b_2 \leq \frac{5\sqrt{3}}{16}} \|x - (\frac{1}{2}, b_2)\|^2 dP + \int_{\Delta_1 \cup \Delta_2} \min_{\frac{\sqrt{3}}{16} \leq b_1 \leq \frac{1}{4}} \|x - (\frac{1}{2}, b_1)\|^2 dP \\ &\geq 0.0901278 > V_2, \end{aligned}$$

which leads to another contradiction. Thus, we see that $\frac{5\sqrt{3}}{16} \leq b_2 \leq \frac{7\sqrt{3}}{16}$. We now show that P -almost surely the Voronoi region of $(\frac{1}{2}, b_1)$ does not contain any point from Δ_3 . For the sake of contradiction, assume that P -almost surely the Voronoi region of $(\frac{1}{2}, b_1)$ contains points from Δ_3 . Then, $\frac{1}{2}(b_1 + b_2) > \frac{\sqrt{3}}{4}$ which implies $b_1 > \frac{\sqrt{3}}{2} - b_2 \geq \frac{\sqrt{3}}{2} - \frac{7\sqrt{3}}{16} = \frac{5\sqrt{3}}{16}$, i.e., $\frac{3\sqrt{3}}{16} < b_1 \leq \frac{\sqrt{3}}{4}$. Then,

$$\begin{aligned} & \int \min_{c \in \alpha} \|x - c\|^2 dP \\ & \geq \int_{\Delta_{33}} \|x - a(33)\|^2 dP + \int_{\Delta_{313} \cup \Delta_{323}} \|x - a(313, 323)\|^2 dP + \int_{\Delta_1 \cup \Delta_2} \min_{\frac{3\sqrt{3}}{16} < b_1 \leq \frac{\sqrt{3}}{4}} \|x - (\frac{1}{2}, b_1)\|^2 dP \\ & \geq \frac{246219}{2816000} = 0.0874357 > V_2, \end{aligned}$$

which leads to a contradiction. Thus, we can assume that the Voronoi region of $(\frac{1}{2}, b_1)$ does not contain any point from Δ_3 yielding $(\frac{1}{2}, b_1) = a(1, 2) = (\frac{1}{2}, \frac{\sqrt{3}}{16})$ and $(\frac{1}{2}, b_2) = a(3) = (\frac{1}{2}, \frac{3\sqrt{3}}{8})$. Hence, the set $\alpha = \{a(1, 2), a(3)\}$ is an optimal set of two-means with quantization error $V_2 = \frac{117}{1408} = 0.0830966$, which is the proposition. \square

Remark 3.2. The set α in the above proposition is a unique optimal set of two-means.

Let us now prove the following proposition.

Proposition 3.3. *Let α be an optimal set of three-means. Then $\alpha = \{a(1), a(2), a(3)\}$ and $V_3 = \frac{189}{7040} = 0.0268466$, where $a(1) = (\frac{1}{8}, \frac{\sqrt{3}}{16})$, $a(2) = (\frac{7}{8}, \frac{\sqrt{3}}{16})$, and $a(3) = (\frac{1}{2}, \frac{3\sqrt{3}}{8})$. Moreover, the Voronoi region of the point $\alpha_3 \cap \Delta_i$ does not contain any point from Δ_j for all $1 \leq j \neq i \leq 3$.*

Proof. Let us consider the three-point set β given by $\beta = \{a(1), a(2), a(3)\}$. Then, the distortion error is given by

$$\int \min_{a \in \alpha} \|x - a\|^2 dP = \sum_{i=1}^3 \int_{\Delta_i} \|x - a(i)\|^2 dP = \frac{189}{7040} = 0.0268466.$$

Since V_3 is the quantization error for three-means, we have $V_3 \leq 0.0268466$. Let α be an optimal set of three-means. As the optimal points are the centroids of their own Voronoi regions we have $\alpha \subset \Delta$. Write $\alpha := \{(a_i, b_i) : 1 \leq i \leq 3\}$. Since $(\frac{1}{2}, \frac{\sqrt{3}}{4})$ is the centroid of the R-triangle, we have

$$(3) \quad \sum_{i=1}^3 (a_i, b_i) P(M((a_i, b_i)|\alpha)) = (\frac{1}{2}, \frac{\sqrt{3}}{4}).$$

Suppose α does not contain any point from Δ_3 . Then, $b_i < \frac{\sqrt{3}}{4}$ for all $1 \leq i \leq 3$ implying $\sum_{i=1}^3 b_i P(M((a_i, b_i)|\alpha)) < \frac{\sqrt{3}}{4} \sum_{i=1}^3 P(M((a_i, b_i)|\alpha)) = \frac{\sqrt{3}}{4}$, which contradicts (3). So, we can assume that α contains a point from Δ_3 . If α contains only one point from $\Delta \setminus \Delta_3$, due to symmetry we can assume that the point lies on the line $x_1 = \frac{1}{2}$, and so

$$\begin{aligned} & \int \min_{c \in \alpha} \|x - c\|^2 dP \geq \int_{\Delta_1 \cup \Delta_2} \min_{c \in \alpha} \|x - c\|^2 dP \geq \int_{\Delta_1 \cup \Delta_2} \|x - a(1, 2)\|^2 dP \\ & = \frac{423}{7040} = 0.0600852 > V_3, \end{aligned}$$

which leads to a contradiction. Similarly, we can show that if α does not contain any point from $\Delta \setminus \Delta_3$ a contradiction will arise. Thus, we conclude that α contains only one point from Δ_3 and two points from $\Delta \setminus \Delta_3$. Due to symmetry of the R-triangle with respect to the line $x_1 = \frac{1}{2}$, we can assume that the point of $\alpha \cap \Delta_3$ lies on the line $x_1 = \frac{1}{2}$, and the two points

of $\alpha \cap (\Delta \setminus \Delta_3)$, say (a, b) and (c, d) , are symmetrically distributed over the triangle Δ with respect to the line $x_1 = \frac{1}{2}$. Let (a, b) and (c, d) lie to the left and right of the line $x_1 = \frac{1}{2}$ respectively. Notice that $\Delta_1 \subset M((a, b)|\alpha)$, $\Delta_2 \subset M((c, d)|\alpha)$, and the Voronoi regions of (a, b) and (c, d) do not contain any point from Δ_{33} . If P -almost surely the Voronoi region of (a, b) does not contain any point from Δ_{31} , we have $(a, b) = a(1) = (\frac{1}{8}, \frac{\sqrt{3}}{16})$. Notice that the point of Δ_{31} closest to $(\frac{1}{8}, \frac{\sqrt{3}}{16})$ is $S_{31}(0, 0)$. Suppose that P almost surely the Voronoi region of (a, b) contains points from Δ_{31} . Then, for some $k > 1$, may be large enough, we must have $\Delta_1 \cup \Delta_{31^k} \subset M((a, b)|\alpha)$, where 1^k is the word obtained from k times concatenation of 1. Without any loss of generality, for calculation simplicity, take $k = 4$. Then, due to symmetry, we have $\Delta_1 \cup \Delta_{31111} \subset M((a, b)|\alpha)$, $\Delta_2 \cup \Delta_{3222} \subset M((c, d)|\alpha)$. Write $A := \Delta_3 \setminus (\Delta_{31111} \cup \Delta_{3222}) = \Delta_{33} \cup \bigcup_{i=1}^2 \Delta_{3i3} \cup \bigcup_{i=1}^2 \Delta_{3ii3} \cup \bigcup_{i=1}^2 \Delta_{3iii3} \cup \Delta_{312} \cup \Delta_{321} \cup \Delta_{3112} \cup \Delta_{3221} \cup \Delta_{31112} \cup \Delta_{32221}$. Then, the distortion error is

$$\begin{aligned} \int \min_{c \in \alpha} \|x - c\|^2 dP &= 2 \int_{\Delta_1 \cup \Delta_{31111}} \|x - a(1, 31111)\|^2 dP + \int_A \|x - E(X : X \in A)\|^2 dP \\ &= \frac{30315288636117}{1128184938496000} = 0.0268709 > V_3, \end{aligned}$$

which leads to a contradiction. Thus, we can conclude that the Voronoi regions of (a, b) and (c, d) do not contain any point from Δ_3 . Hence, the optimal set of three-means is $\{a(1), a(2), a(3)\}$ and the quantization error is $V_3 = \frac{189}{7040} = 0.0268466$. By finding the perpendicular bisectors of the line segments joining the points in α_3 , we see that the perpendicular bisector of the line segments joining the points $\alpha_3 \cap \Delta_i$ and $\alpha_3 \cap \Delta_j$ does not intersect any of Δ_i or Δ_j for $1 \leq i \neq j \leq 3$. Thus, the Voronoi region of the point $\alpha_3 \cap \Delta_i$ does not contain any point from Δ_j for all $1 \leq j \neq i \leq 3$. Hence, the proof of the proposition is complete. \square

Proposition 3.4. *Let α_n be an optimal set of n -means for all $n \geq 3$. Then, (i) $\alpha_n \cap \Delta_i \neq \emptyset$ for all $1 \leq i \leq 3$, (ii) α_n does not contain any point from $\Delta \setminus (\Delta_1 \cup \Delta_2 \cup \Delta_3)$, and (iii) the Voronoi region of any points in $\alpha_n \cap \Delta_i$ does not contain any point from Δ_j for all $1 \leq j \neq i \leq 3$.*

Proof. Let α_n be an optimal set of n -means for $n \geq 3$. By Proposition 3.3, we see that the proposition is true for $n = 3$. We now show that the proposition is true for $n \geq 4$. Consider the set of four points $\beta := \{a(1), a(2), a(31, 32), a(33)\}$. Since V_n is the quantization error for n -means for $n \geq 4$, we have

$$V_n \leq V_4 \leq \int \min_{b \in \beta} \|x - b\|^2 dP = \frac{459}{28160} = 0.0162997.$$

If α_n does not contain any point from Δ_3 , then

$$V_n \geq \int_{\Delta_{33}} \min_{(a,b) \in \alpha} \|(x_1, x_2) - (a, b)\|^2 dP \geq \left\| \left(\frac{1}{2}, \frac{3\sqrt{3}}{8} \right) - \left(\frac{1}{2}, \frac{\sqrt{3}}{4} \right) \right\|^2 P(\Delta_{33}) = \frac{27}{1600},$$

implying $V_n \geq \frac{27}{1600} = 0.016875 > V_n$, which leads to a contradiction. So, we can assume that $\alpha_n \cap \Delta_3 \neq \emptyset$. If $\alpha_n \subset \Delta_3$, then

$$V_n \geq 2 \int_{\Delta_1} \min_{(a,b) \in \alpha_n} \|(x_1, x_2) - (a, b)\|^2 dP \geq 2 \left\| S_1\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) - S_3(0, 0) \right\|^2 P(\Delta_1) = \frac{1}{40} = 0.025 > V_n.$$

which gives a contradiction. So, we can assume that α_n contains points below the horizontal line $x_2 = \frac{\sqrt{3}}{4}$. If α_n contains only one point below the line $x_2 = \frac{\sqrt{3}}{4}$, then due to symmetry the

point must lie on the line $x_1 = \frac{1}{2}$, and so

$$V_n \geq 2 \int_{\Delta_{133} \cup \Delta_{131}} \|(x_1, x_2) - S_3(0, 0)\|^2 dP + \int_{\Delta_{12} \cup \Delta_{21}} \|(x_1, x_2) - a(12, 21)\|^2 dP = 0.0233299 > V_n,$$

which is a contradiction. So, we can assume that α_n contains at least two points below the line $x_2 = \frac{\sqrt{3}}{4}$, and then due to symmetry between the two points, one point will belong to Δ_1 and one point will belong to Δ_2 . Thus, we see that $\alpha_n \cap \Delta_i \neq \emptyset$ for all $1 \leq i \leq 3$, which completes the proof of (i). We now show that α_n does not contain any point from $\Delta \setminus (\Delta_1 \cup \Delta_2 \cup \Delta_3)$. If α_n contains only one point from $\Delta \setminus (\Delta_1 \cup \Delta_2 \cup \Delta_3)$, then due to symmetry the point must lie on the line $x_1 = \frac{1}{2}$, but as α_n contains points from both Δ_1 and Δ_2 , the Voronoi region of any point on the line $x_1 = \frac{1}{2}$ can not contain any point from $\Delta_1 \cup \Delta_2$, which leads to a contradiction. If α_n contains two points from $\Delta \setminus (\Delta_1 \cup \Delta_2 \cup \Delta_3)$, then due to symmetry quantization error can be strictly reduced by moving one point to Δ_1 and one point to Δ_2 . If α_n contains three or more points from $\Delta \setminus (\Delta_1 \cup \Delta_2 \cup \Delta_3)$, by redistributing the points among Δ_i for $1 \leq i \leq 3$, the quantization error can be strictly reduced. Thus, α_n does not contain any point from $\Delta \setminus (\Delta_1 \cup \Delta_2 \cup \Delta_3)$ yielding the proof of (ii). Since $n \geq 3$, for any $(a, b) \in \alpha_n \cap \Delta_i$, the Voronoi region of (a, b) is contained in the Voronoi region of $\alpha_3 \cap \Delta_i$, and by Proposition 3.3, the Voronoi region of $\alpha_3 \cap \Delta_i$ does not contain any point from Δ_j for $1 \leq j \neq i \leq 3$, we can say that the Voronoi region of the point from $\alpha_n \cap \Delta_i$ does not contain any point from Δ_j for $1 \leq j \neq i \leq 3$ which is (iii). Thus, the proof of the proposition is complete. \square

The following lemma is also true here.

Lemma 3.5. (see [CR, Lemma 3.7]) Let $P = \sum_{\omega \in I^k} \frac{1}{3^k} P \circ S_\omega^{-1}$ for some $k \geq 1$. Let α be an optimal set of n -means for the R -measure P . Then, $\{S_\omega(a) : a \in \alpha\}$ is an optimal set of n -means for the image measure $P \circ S_\omega^{-1}$. The converse is also true: If β is an optimal set of n -means for the image measure $P \circ S_\omega^{-1}$, then $\{S_\omega^{-1}(a) : a \in \beta\}$ is an optimal set of n -means for P .

Proposition 3.6. Let α_n be an optimal set of n -means for $n \geq 3$. Then, for $c \in \alpha_n$ either $c = a(\omega)$ or $c = a(\omega 1, \omega 2)$ for some $\omega \in I^*$.

Proof. Let α_n be an optimal set of n -means for $n \geq 3$ and $c \in \alpha_n$. Then, by Proposition 3.4, we see that either $c \in \alpha_n \cap \Delta_i$ for some $1 \leq i \leq 3$. Without any loss of generality, we can assume that $c \in \alpha_n \cap \Delta_1$. If $\text{card}(\alpha_n \cap \Delta_1) = 1$, then by Lemma 3.5, $S_1^{-1}(\alpha_n \cap \Delta_1)$ is an optimal set of one-mean yielding $c = S_1(\frac{1}{2}, \frac{\sqrt{3}}{4}) = a(1)$. If $\text{card}(\alpha_n \cap \Delta_1) = 2$, then by Lemma 3.5, $S_1^{-1}(\alpha_n \cap \Delta_1)$ is an optimal set of two-means, i.e., $S_1^{-1}(\alpha_n \cap \Delta_1) = \{a(1, 2), a(3)\}$ yielding $c = a(11, 12)$ or $c = a(13)$. Similarly, if $\text{card}(\alpha_n \cap \Delta_1) = 3$, then $c = a(11), a(12)$, or $c = a(13)$. Let $\text{card}(\alpha_n \cap \Delta_1) \geq 4$. Then, as similarity mappings preserve the ratio of the distances of a point from any other two points, using Proposition 3.4 again, we have $(\alpha_n \cap \Delta_1) \cap \Delta_{1i} = \alpha_n \cap \Delta_{1i} \neq \emptyset$ for $1 \leq i \leq 3$, and $\alpha_n \cap \Delta_1 = \cup_{i=1}^3 (\alpha_n \cap \Delta_{1i})$. Without any loss of generality assume that $c \in \alpha_n \cap \Delta_{11}$. If $\text{card}(\alpha_n \cap \Delta_{11}) = 1$, then $c = a(11)$. If $\text{card}(\alpha_n \cap \Delta_{11}) = 2$, then $c = a(111, 112)$ or $c = a(113)$. If $\text{card}(\alpha_n \cap \Delta_{11}) = 3$, then $c = a(111), a(112)$, or $c = a(113)$. If $\text{card}(\alpha_n \cap \Delta_{11}) \geq 4$, then proceeding inductively in the similar way, we can find a word $\omega \in I^*$ with $11 \prec \omega$, such that $c \in \alpha_n \cap \Delta_\omega$. If $\text{card}(\alpha_n \cap \Delta_\omega) = 1$, then $c = a(\omega)$. If $\text{card}(\alpha_n \cap \Delta_\omega) = 2$, then $c = a(\omega 1, \omega 2)$ or $a(\omega 3)$. If $\text{card}(\alpha_n \cap \Delta_\omega) = 3$, then $c = a(\omega 1), a(\omega 2)$, or $a(\omega 3)$. Thus, the proof of the proposition is yielded. \square

Note 3.7. Let α be an optimal set of n -means for some $n \geq 2$. Then, by Proposition 3.6, for $a \in \alpha$ we have P -almost surely, $M(a|\alpha) = \Delta_\omega$ if $a = a(\omega)$, and $M(a|\alpha) = \Delta_{\omega 1} \cup \Delta_{\omega 2}$ if

$a = a(\omega 1, \omega 2)$. For $\omega \in I^*$, write

$$(4) \quad E(\omega) := \int_{\Delta_\omega} \|x - a(\omega)\|^2 dP \text{ and } E(\omega 1, \omega 2) := \int_{\Delta_{\omega 1} \cup \Delta_{\omega 2}} \|x - a(\omega 1, \omega 2)\|^2 dP.$$

Let us now give the following lemma.

Lemma 3.8. *For any $\omega \in I^*$, let $E(\omega)$ and $E(\omega 1, \omega 2)$ be defined by (4). Then, $E(\omega 1, \omega 2) = \frac{47}{18}E(\omega 3) = \frac{47}{120}E(\omega)$, and $E(\omega 1) = E(\omega 2) = \frac{1}{12}E(\omega 3) = \frac{1}{80}E(\omega)$.*

Proof. By (2), we have

$$\begin{aligned} E(\omega 1, \omega 2) &= \int_{\Delta_{\omega 1} \cup \Delta_{\omega 2}} \|x - a(\omega 1, \omega 2)\|^2 dP = \int_{\Delta_{\omega 1}} \|x - a(\omega 1, \omega 2)\|^2 dP + \int_{\Delta_{\omega 2}} \|x - a(\omega 1, \omega 2)\|^2 dP \\ &= p_{\omega 1}(s_{\omega 1}^2 V + \|a(\omega 1) - a(\omega 1, \omega 2)\|^2) + p_{\omega 2}(s_{\omega 2}^2 V + \|a(\omega 2) - a(\omega 1, \omega 2)\|^2). \end{aligned}$$

Notice that

$$\begin{aligned} \|a(\omega 1) - a(\omega 1, \omega 2)\|^2 &= \|S_{\omega 1}(\frac{1}{2}, \frac{\sqrt{3}}{4}) - \frac{1}{2}(S_{\omega 1}(\frac{1}{2}, \frac{\sqrt{3}}{4}) + S_{\omega 2}(\frac{1}{2}, \frac{\sqrt{3}}{4}))\|^2 \\ &= \frac{1}{4}\|S_{\omega 1}(\frac{1}{2}, \frac{\sqrt{3}}{4}) - S_{\omega 2}(\frac{1}{2}, \frac{\sqrt{3}}{4})\|^2 = \frac{1}{4}s_\omega^2\|S_1(\frac{1}{2}, \frac{\sqrt{3}}{4}) - S_2(\frac{1}{2}, \frac{\sqrt{3}}{4})\|^2 = \frac{9}{64}s_\omega^2, \end{aligned}$$

and similarly, $\|a(\omega 2) - a(\omega 1, \omega 2)\|^2 = \frac{9}{64}s_\omega^2$. Thus, we obtain,

$$\begin{aligned} E(\omega 1, \omega 2) &= p_{\omega 1}(s_{\omega 1}^2 V + \frac{9}{64}s_\omega^2) + p_{\omega 2}(s_{\omega 2}^2 V + \frac{9}{64}s_\omega^2) = p_\omega s_\omega^2 V(p_1 s_1^2 + p_2 s_2^2) + \frac{9}{64}p_\omega s_\omega^2(p_1 + p_2) \\ \text{yielding } E(\omega 1, \omega 2) &= p_\omega s_\omega^2 V(p_1 s_1^2 + p_2 s_2^2 + \frac{9}{160}\frac{1}{V}) = p_\omega s_\omega^2 V \frac{47}{120} = \frac{47}{120}E(\omega). \text{ Since } p_1 = p_2, s_1 = s_2, \\ \text{we have } E(\omega 1) &= p_{\omega 1}s_{\omega 1}^2 V = \frac{1}{80}p_\omega s_\omega^2 V = E(\omega 2). \text{ Again, } E(\omega 3) = p_{\omega 3}s_{\omega 3}^2 V = E(\omega)p_3 s_3^2 = \frac{3}{20}E(\omega). \text{ Hence,} \end{aligned}$$

$$E(\omega 1, \omega 2) = \frac{47}{18}E(\omega 3) = \frac{47}{120}E(\omega) \text{ and } E(\omega 1) = E(\omega 2) = \frac{1}{12}E(\omega 3) = \frac{1}{80}E(\omega),$$

which is the lemma. \square

The following lemma plays an important role to prove the main theorem of the paper.

Lemma 3.9. *Let $\omega, \tau \in I^*$. Then*

- (i) $E(\omega) > E(\tau)$ if and only if $E(\omega 1, \omega 2) + E(\omega 3) + E(\tau) < E(\omega) + E(\tau 1, \tau 2) + E(\tau 3)$;
- (ii) $E(\omega) > E(\tau 1, \tau 2)$ if and only if $E(\omega 1, \omega 2) + E(\omega 3) + E(\tau 1, \tau 2) < E(\omega) + E(\tau 1) + E(\tau 2)$;
- (iii) $E(\omega 1, \omega 2) > E(\tau)$ if and only if $E(\omega 1) + E(\omega 2) + E(\tau) < E(\omega 1, \omega 2) + E(\tau 1, \tau 2) + E(\tau 3)$;
- (iv) $E(\omega 1, \omega 2) > E(\tau 1, \tau 2)$ if and only if $E(\omega 1) + E(\omega 2) + E(\tau 1, \tau 2) < E(\omega 1, \omega 2) + E(\tau 1) + E(\tau 2)$;

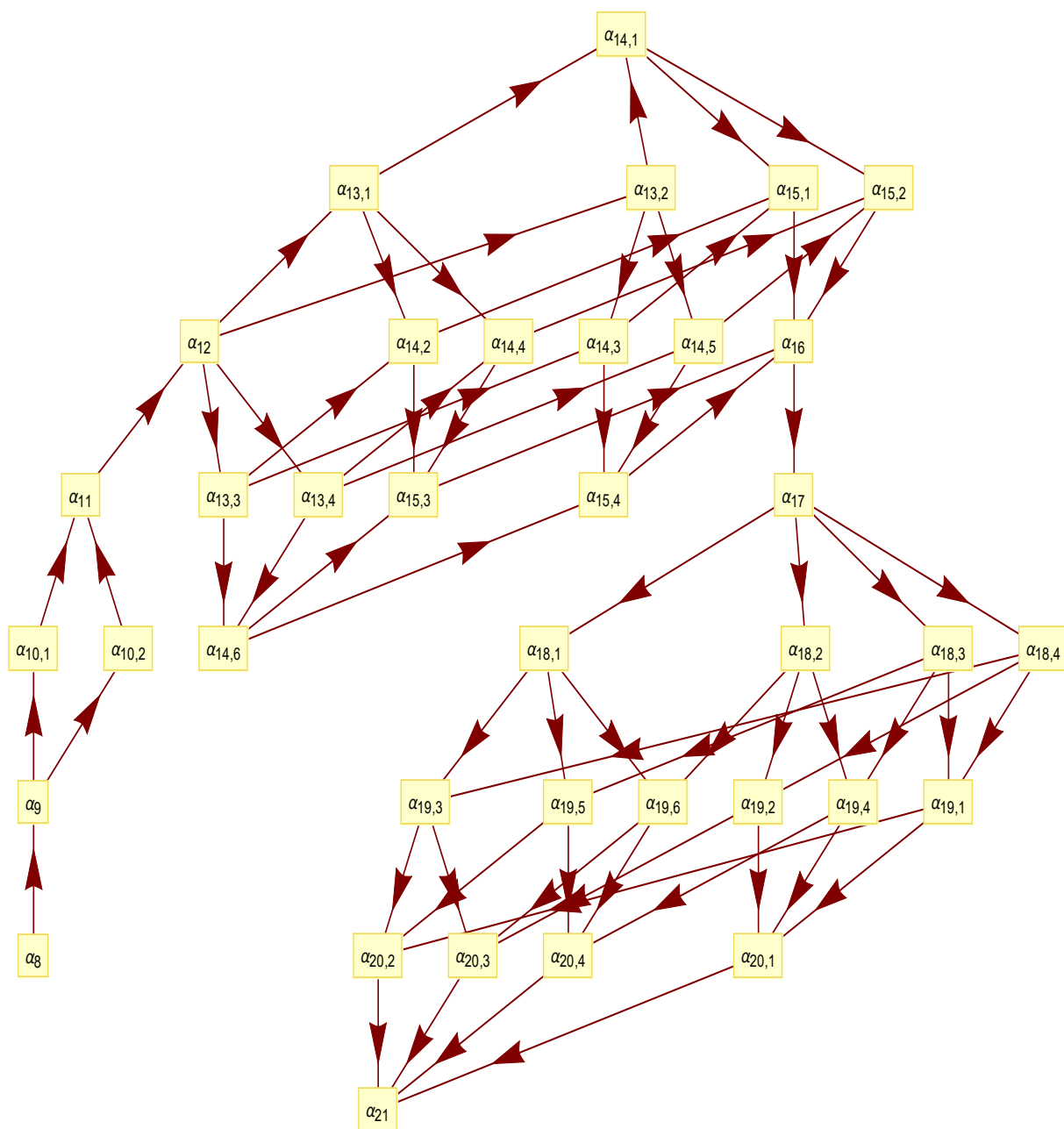
where for any $\omega \in I^*$, $E(\omega)$ and $E(\omega 1, \omega 2)$ are defined by (4).

Proof. To prove (i), using Lemma 3.8, we see that

$$\begin{aligned} LHS &= E(\omega 1, \omega 2) + E(\omega 3) + E(\tau) = (\frac{47}{120} + \frac{3}{20})E(\omega) + E(\tau) = \frac{13}{24}E(\omega) + E(\tau), \\ RHS &= E(\omega) + E(\tau 1, \tau 2) + E(\tau 3) = E(\omega) + \frac{13}{24}E(\tau). \end{aligned}$$

Thus, $LHS < RHS$ if and only if $\frac{13}{24}E(\omega) + E(\tau) < E(\omega) + \frac{13}{24}E(\tau)$, which yields $E(\tau) < E(\omega)$. Thus (i) is proved. Proceeding in the similar way, (ii), (iii) and (iv) can be proved. Thus, the lemma is deduced. \square

In the following theorem, we give the induction formula to determine the optimal sets of n -means for any $n \geq 2$.



Theorem 3.10. *For any $n \geq 2$, let $\alpha_n := \{a(i) : 1 \leq i \leq n\}$ be an optimal set of n -means, i.e., $\alpha_n \in \mathcal{C}_n := \mathcal{C}_n(P)$. For $\omega \in I^*$, let $E(\omega)$ and $E(\omega 1, \omega 2)$ be defined by (4). Set*

$$\tilde{E}(a(i)) := \begin{cases} E(\omega) & \text{if } a(i) = a(\omega) \text{ for some } \omega \in I^*, \\ E(\omega 1, \omega 2) & \text{if } a(i) = a(\omega 1, \omega 2) \text{ for some } \omega \in I^*, \end{cases}$$

and $W(\alpha_n) := \{a(j) : a(j) \in \alpha_n \text{ and } \tilde{E}(a(j)) \geq \tilde{E}(a(i)) \text{ for all } 1 \leq i \leq n\}$. Take any $a(j) \in W(\alpha_n)$, and write

$$\alpha_{n+1}(a(j)) := \begin{cases} (\alpha_n \setminus \{a(j)\}) \cup \{a(\omega 1, \omega 2), a(\omega 3)\} & \text{if } a(j) = a(\omega), \\ (\alpha_n \setminus \{a(j)\}) \cup \{a(\omega 1), a(\omega 2)\} & \text{if } a(j) = a(\omega 1, \omega 2). \end{cases}$$

Then $\alpha_{n+1}(a(j))$ is an optimal set of $(n+1)$ -means, and the number of such sets is given by

$$\text{card}\left(\bigcup_{\alpha_n \in \mathcal{C}_n} \{\alpha_{n+1}(a(j)) : a(j) \in W(\alpha_n)\}\right).$$

Proof. By Proposition 3.1 and Proposition 3.3, we know that the optimal sets of two- and three-means are $\{a(1, 2), a(3)\}$ and $\{a(1), a(2), a(3)\}$. Notice that by Lemma 3.8, we know $E(1, 2) > E(3)$. Hence, the theorem is true for $n = 2$. For any $n \geq 2$, let us now assume that α_n is an optimal set of n -means. Let $\alpha_n := \{a(i) : 1 \leq i \leq n\}$. Let $\tilde{E}(a(i))$ and $W(\alpha_n)$ be defined as in the hypothesis. If $a(j) \notin W(\alpha_n)$, i.e., if $a(j) \in \alpha_n \setminus W(\alpha_n)$, then by Lemma 3.9, the error

$$\sum_{a(i) \in (\alpha_n \setminus \{a(j)\})} E(a(i)) + E(\omega 1, \omega 2) + E(\omega 3) \text{ if } a(j) = a(\omega),$$

or

$$\sum_{a(i) \in (\alpha_n \setminus \{a(j)\})} E(a(i)) + E(\omega 1) + E(\omega 2) \text{ if } a(j) = a(\omega 1, \omega 2),$$

obtained in this case is strictly greater than the corresponding error obtained in the case when $a(j) \in W(\alpha_n)$. Hence, for any $a(j) \in W(\alpha_n)$, the set $\alpha_{n+1}(a(j))$, where

$$\alpha_{n+1}(a(j)) := \begin{cases} (\alpha_n \setminus \{a(j)\}) \cup \{a(\omega 1, \omega 2), a(\omega 3)\} & \text{if } a(j) = a(\omega), \\ (\alpha_n \setminus \{a(j)\}) \cup \{a(\omega 1), a(\omega 2)\} & \text{if } a(j) = a(\omega 1, \omega 2). \end{cases}$$

is an optimal set of $(n+1)$ -means, and the number of such sets is

$$\text{card}\left(\bigcup_{\alpha_n \in \mathcal{C}_n} \{\alpha_{n+1}(a(j)) : a(j) \in W(\alpha_n)\}\right).$$

Thus, the proof of the theorem is complete. \square

Remark 3.11. Once an optimal set of n -means is known, by using (2), the corresponding quantization error can easily be calculated.

Using the induction formula given by Theorem 3.10, we obtain some results and observations about the optimal sets of n -means, which are given in the following section.

4. SOME RESULTS AND OBSERVATIONS

First, we explain about some notations that we are going to use in this section. Recall that the optimal set of one-mean consists of the expected value of the random variable X , and the corresponding quantization error is its variance. Let α_n be an optimal set of n -means, i.e., $\alpha_n \in \mathcal{C}_n$, and then for any $a \in \alpha_n$, we have $a = a(\omega)$, or $a = a(\omega 1, \omega 2)$ for some $\omega \in I^k$, $k \geq 1$. For any $n \geq 2$, if $\text{card}(\mathcal{C}_n) = k$, we write

$$\mathcal{C}_n = \begin{cases} \{\alpha_{n,1}, \alpha_{n,2}, \dots, \alpha_{n,k}\} & \text{if } k \geq 2, \\ \{\alpha_n\} & \text{if } k = 1. \end{cases}$$

If $\text{card}(\mathcal{C}_n) = k$ and $\text{card}(\mathcal{C}_{n+1}) = m$, then either $1 \leq k \leq m$, or $1 \leq m \leq k$ (see Table 1). Moreover, by Theorem 3.10, an optimal set at stage n can contribute multiple distinct optimal sets at stage $n+1$, and multiple distinct optimal sets at stage n can contribute one common optimal set at stage $n+1$; for example from Table 1, one can see that the number of $\alpha_{12} = 1$, the number of $\alpha_{13} = 4$, the number of $\alpha_{14} = 6$, the number of $\alpha_{15} = 4$, and the number of $\alpha_{16} = 1$.

n	$\text{card}(\mathcal{C}_n)$	n	$\text{card}(\mathcal{C}_n)$	n	$\text{card}(\mathcal{C}_n)$	n	$\text{card}(\mathcal{C}_n)$	n	$\text{card}(\mathcal{C}_n)$	n	$\text{card}(\mathcal{C}_n)$
5	1	18	4	31	6	44	1	57	495	70	56
6	1	19	6	32	4	45	8	58	792	71	28
7	2	20	4	33	1	46	28	59	924	72	8
8	1	21	1	34	6	47	56	60	792	73	1
9	1	22	1	35	15	48	70	61	495	74	1
10	2	23	6	36	20	49	56	62	220	75	12
11	1	24	15	37	15	50	28	63	66	76	66
12	1	25	20	38	6	51	8	64	12	77	220
13	4	26	15	39	1	52	1	65	1	78	495
14	6	27	6	40	1	53	1	66	8	79	792
15	4	28	1	41	4	54	12	67	28	80	924
16	1	29	1	42	6	55	66	68	56	81	792
17	1	30	4	43	4	56	220	69	70	82	495

TABLE 1. Number of α_n in the range $5 \leq n \leq 82$.

By $\alpha_{n,i} \rightarrow \alpha_{n+1,j}$, it is meant that the optimal set $\alpha_{n+1,j}$ at stage $n+1$ is obtained from the optimal set $\alpha_{n,i}$ at stage n , similar is the meaning for the notations $\alpha_n \rightarrow \alpha_{n+1,j}$, or $\alpha_{n,i} \rightarrow \alpha_{n+1}$, for example from Figure 3:

$$\begin{aligned} & \{\alpha_{12} \rightarrow \alpha_{13,1}, \alpha_{12} \rightarrow \alpha_{13,2}, \alpha_{12} \rightarrow \alpha_{13,3}, \alpha_{12} \rightarrow \alpha_{13,4}\}, \\ & \{\{\alpha_{13,1} \rightarrow \alpha_{14,1}, \alpha_{13,1} \rightarrow \alpha_{14,2}, \alpha_{13,1} \rightarrow \alpha_{14,4}\}, \{\alpha_{13,2} \rightarrow \alpha_{14,1}, \alpha_{13,2} \rightarrow \alpha_{14,3}, \alpha_{13,2} \rightarrow \alpha_{14,5}\}, \\ & \{\alpha_{13,3} \rightarrow \alpha_{14,2}, \alpha_{13,3} \rightarrow \alpha_{14,3}, \alpha_{13,3} \rightarrow \alpha_{14,6}\}, \{\alpha_{13,4} \rightarrow \alpha_{14,4}, \alpha_{13,4} \rightarrow \alpha_{14,5}, \alpha_{13,4} \rightarrow \alpha_{14,6}\}\}. \end{aligned}$$

Moreover, we see that

$$\begin{aligned} \alpha_6 &= \{a(1), a(2), a(31), a(32), a(333), a(331, 332)\} \text{ with } V_6 = \frac{3537}{563200} = 0.00628018; \\ \alpha_{7,1} &= \{a(1), a(23), a(21, 22), a(31), a(32), a(333), a(331, 332)\}; \\ \alpha_{7,2} &= \{a(13), a(11, 12), a(2), a(31), a(32), a(333), a(331, 332)\} \\ & \text{with } V_7 = \frac{1521}{281600} = 0.00540128; \\ \alpha_8 &= \{a(13), a(11, 12), a(23), a(21, 22), a(31), a(32), a(333), a(331, 332)\} \\ & \text{with } V_8 = \frac{2547}{563200} = 0.00452237; \\ \alpha_9 &= \{a(13), a(11, 12), a(23), a(21, 22), a(31), a(32), a(333), a(331), a(332)\} \\ & \text{with } V_9 = \frac{9171}{2816000} = 0.00325675; \\ \alpha_{10,1} &= \{a(13), a(11, 12), a(23), a(21), a(22), a(31), a(32), a(333), a(331), a(332)\}; \\ \alpha_{10,2} &= \{a(13), a(11), a(12), a(23), a(21, 22), a(31), a(32), a(333), a(331), a(332)\} \\ & \text{with } V_{10} = \frac{7191}{2816000} = 0.00255362; \\ \alpha_{11} &= \{a(13), a(11), a(12), a(23), a(21), a(22), a(31), a(32), a(333), a(331), a(332)\} \\ & \text{with } V_{11} = \frac{5211}{2816000} = 0.0018505; \end{aligned}$$

and so on.

Remark 4.1. By Theorem 3.10, we note that to obtain an optimal set of $(n + 1)$ -means one needs to know an optimal set of n -means. Unlike the probability distribution supported by the classical R-triangle (see [CR]), for the probability distribution supported by the nonhomogeneous R-triangle considered in this paper, to obtain the optimal sets of n -means a closed formula is not known yet.

REFERENCES

- [AW] E.F. Abaya and G.L. Wise, *Some remarks on the existence of optimal quantizers*, Statistics & Probability Letters, Volume 2, Issue 6, December 1984, pp. 349-351.
- [CR] D. Çömez and M.K. Roychowdhury, *Quantization of probability distributions on R-triangles*, arXiv:1605.09701 [math.DS].
- [DFG] Q. Du, V. Faber and M. Gunzburger, *Centroidal Voronoi Tessellations: Applications and Algorithms*, SIAM Review, Vol. 41, No. 4 (1999), 637-676.
- [DR] C.P. Dettmann and M.K. Roychowdhury, *Quantization for uniform distributions on equilateral triangles*, Real Analysis Exchange, Vol. 42(1), 2017, pp. 149-166.
- [GG] A. Gersho and R.M. Gray, *Vector quantization and signal compression*, Kluwer Academy publishers: Boston, 1992.
- [GKL] R.M. Gray, J.C. Kieffer and Y. Linde, *Locally optimal block quantizer design*, Information and Control, 45 (1980), 178-198.
- [GL1] A. Györfy and T. Linder, *On the structure of optimal entropy-constrained scalar quantizers*, IEEE transactions on information theory, vol. 48, no. 2, February 2002.
- [GL2] S. Graf and H. Luschgy, *Foundations of quantization for probability distributions*, Lecture Notes in Mathematics 1730, Springer, Berlin, 2000.
- [GL3] S. Graf and H. Luschgy, *The Quantization of the Cantor Distribution*, Math. Nachr., 183 (1997), 113-133.
- [GN] R. Gray and D. Neuhoff, *Quantization*, IEEE Trans. Inform. Theory, 44 (1998), 2325-2383.
- [MR] M. Morán and J. Rey, *Geometry of self-similar measures*, Annales Academiae Scientiarum Fennicae Mathematica, Vol. 22, 1997, 365-386.
- [R] M.K. Roychowdhury, *Quantization and centroidal Voronoi tessellations for probability measures on dyadic Cantor sets*, arXiv:1509.06037 [math.DS].
- [Z] R. Zam, *Lattice Coding for Signals and Networks: A Structured Coding Approach to Quantization, Modulation, and Multiuser Information Theory*, Cambridge University Press, 2014.

SCHOOL OF MATHEMATICAL AND STATISTICAL SCIENCES, UNIVERSITY OF TEXAS RIO GRANDE VALLEY,
1201 WEST UNIVERSITY DRIVE, EDINBURG, TX 78539-2999, USA.

E-mail address: mrinal.roychowdhury@utrgv.edu